Different Views of Information
Part II

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Geometry in Cryptography and Communication

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random string

which of these strings is random?

- 0000000000000000000000000000000000000000000000000
- 0101010101010101010101010101010101010101010101010
- 0011111001001101100111011000001110101011011110011
- 1010110111111000010101000101100010100010101110110
- 1100100100001111111011010101000100010000101101000

random string

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- 0000000000000000000000000000000000000000000000000
- 0101010101010101010101010101010101010101010101010
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- $e \approx 10.1011011111100001010100010110001010001010111011 01$
- $\pi \approx 11.0010010000111111011010101000100010000101101000 11$

Explain your guess!

measure of information: Kolmogorov's approach

Informal Definition: Kolmogorov complexity:

$\mathrm{C}(\mathbf{x}) = \min\{|p| \ : \ \text{program } p \text{ produced } \mathbf{x}\}$

$\mathrm{C}(\mathbf{x}|\mathbf{y}) = \min\{|p| \ : \ \text{program } p \text{ produced } \mathbf{x} \text{ given } \mathbf{y}\}$

measure of information: Kolmogorov's approach

Formal Definition: Kolmogorov complexity (with a *decompressor* U)

$C_U(\mathbf{x}) = \min\{|p| \ : \ U(p) = \mathbf{x}\}$

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$C_U(\mathbf{x}) = \min\{|p| \ : \ U(p) = \mathbf{x}\}$

Theorem

There exists a (computable) U such that for every other V and all strings $\mathbf{x}$,

$$C_U(\mathbf{x}) \leq C_V(\mathbf{x}) + O(1).$$

measure of information: Kolmogorov's approach

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Theorem

There exists a (computable) U such that for every other V there is a constant λ such that for all strings x

$$C_U(\mathbf{x}) \leq C_V(\mathbf{x}) + \lambda.$$

Proof.

Idea: Let U be a universal machine that can simulate any other machine V... □

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In what follows we fix U and let $\mathrm{C}(\mathbf{x}) := C_U(\mathbf{x})$

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- there exists a $\lambda \geq$ s.t. for every n, at least 99% of strings of length n satisfy $n - \lambda \ \leq \ \mathrm{C}(\mathbf{x}) \ \leq \ n + \lambda$.

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- for every n there exists a string of length n with complexity at least n,
- there exists a $\lambda \geq$ s.t. for every n, at least 99% of strings of length n satisfy $n - \lambda \ \leq \ \mathrm{C}(\mathbf{x}) \ \leq \ n + \lambda$.
- if there is an algorithm that for every input n produces a list of strings S_n, [binary expansion of n] $\mapsto$ `alg` $\mapsto$ [list of elements of S_n]
  then
  - for every $\mathbf{x} \in S_n$ we have $\mathrm{C}(\mathbf{x}) \leq \log \mathrm{Card}(S_n) + O(\log n)$
  - for most $\mathbf{x} \in S_n$ we have $\mathrm{C}(\mathbf{x}) \geq \log \mathrm{Card}(S_n) - O(1)$

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another simple property:

- if a bit string $\mathbf{x}$ contains $p_0 n$ zeros and $p_1 n$ ones, then

  $\mathrm{C}(\mathbf{x}) \leq \log \frac{n!}{(p_0 n)! \cdot (p_1 n)!} + O(\log n)$

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  $\mathrm{C}(\mathbf{x}) \leq \log \frac{n!}{(p_0 n)! \cdot (p_1 n)!} + O(\log n) = \left(p_0 \log \frac{1}{p_0} + p_1 \log \frac{1}{p_1} \right) n + O(\log n)$

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terminology: $\mathbf{x}$ is λ-**incompressible** if $\mathrm{C}(\mathbf{x}) \geq |\mathbf{x}| - \lambda$

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informal wording: $\mathbf{x}$ is **random** if $\mathrm{C}(\mathbf{x}) \approx |\mathbf{x}|$

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informal wording: $\mathbf{x}$ is **random** if $\mathrm{C}(\mathbf{x}) \approx |\mathbf{x}|$

intuition: $\mathbf{x}$ is **random** if there is no regularities in it

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simple properties of conditional Kolmogorov complexity:

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- $\mathrm{C}(\mathbf{x} \mid \mathbf{y}) \leq \mathrm{C}(\mathbf{x}) + O(1)$,

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- $\mathrm{C}(\mathbf{x} \mid \mathbf{x}) \leq O(1)$,

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- $\mathrm{C}(\mathbf{x} \mid \mathbf{y}) \leq \mathrm{C}(\mathbf{x}) + O(1)$,
- $\mathrm{C}(\mathbf{x} \mid \mathbf{x}) \leq O(1)$,
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- Kolmogorov complexity $\mathrm{C}(\mathbf{x})$ is not a computable function

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strange properties:

- Kolmogorov complexity $\mathrm{C}(\mathbf{x})$ is not a computable function
- no algorithm which, given n, produces a string $\mathbf{x}_n$ such that $\mathrm{C}(\mathbf{x}_n) > n$

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strange properties:

- Kolmogorov complexity $\mathrm{C}(\mathbf{x})$ is not a computable function
- no algorithm which, given n, produces a string $\mathbf{x}_n$ such that $\mathrm{C}(\mathbf{x}_n) > n$
- almost all statements of the form "$\mathrm{C}(\mathbf{x}) > |\mathbf{x}|/2$" are unprovable
  (a sort of Gödel's *incompleteness* theorem)

secure secret key: Kolmogorov's approach

what is a secure secret key of size n?

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possible answer: **x** is secure if any *simple* device brute-forcing the keys would try this **x** at step $\sim 2^n$.

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secure secret key: Kolmogorov's approach

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possible answer: **x** is secure if any *simple* device brute-forcing the keys would try this **x** at step $\sim 2^n$.

this is essentially equivalent to the condition $\mathrm{C}(\mathbf{x}) \approx n$ (i.e., **x** is random)

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Definition of Kolmogorov complexity:

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Information theory:

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Information theory:

- $\mathrm{C}(\mathbf{x}, \mathbf{y}) \leq \mathrm{C}(\mathbf{x}) + \mathrm{C}(\mathbf{y}) + O(\log(|\mathbf{x}| + |\mathbf{y}|))$

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- $\mathrm{C}(\mathbf{x}, \mathbf{y}) = \mathrm{C}(\mathbf{x}) + \mathrm{C}(\mathbf{y} \mid \mathbf{x}) \pm O(\log(|\mathbf{x}| + |\mathbf{y}|))$ [Kolmogorov–Levin]

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Definition: mutual information

$\mathrm{I}(\mathbf{x} : \mathbf{y}) := \mathrm{C}(\mathbf{y}) - \mathrm{C}(\mathbf{y} \mid \mathbf{x})$

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measure of information: Kolmogorov's approach

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Symmetry of the mutual information:

- $\begin{aligned}\mathrm{I}(\mathbf{x} : \mathbf{y}) &= \mathrm{C}(\mathbf{x}) + \mathrm{C}(\mathbf{y}) - \mathrm{C}(\mathbf{x},\mathbf{y}) \pm O(\log(|\mathbf{x}| + |\mathbf{y}|)) \\ &= \mathrm{I}(\mathbf{y} : \mathbf{x}) \pm O(\log(|\mathbf{x}| + |\mathbf{y}|))\end{aligned}$

Kolmogorov complexity: information inequalities

similar to Shannon's case:

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- monotonicity:
  $\mathrm{C}(\mathbf{x}) \leq \mathrm{C}(\mathbf{x}, \mathbf{y}) + O(1)$, cf. $\mathrm{C}(\mathbf{y} \mid \mathbf{x}) \geq 0$

Kolmogorov complexity: information inequalities

similar to Shannon's case:

- monotonicity:
  $\mathrm{C}(\mathbf{x}) \leq \mathrm{C}(\mathbf{x}, \mathbf{y}) + O(1)$, cf. $\mathrm{C}(\mathbf{y} \mid \mathbf{x}) \geq 0$
- subadditivity:
  $\mathrm{C}(\mathbf{x}, \mathbf{y}) \leq \mathrm{C}(\mathbf{x}) + \mathrm{C}(\mathbf{y}) + O(\log(|\mathbf{x}| + |\mathbf{y}|))$, cf. $\mathrm{I}(\mathbf{x} : \mathbf{y}) \geq -O(\log(|\mathbf{x}| + |\mathbf{y}|))$

Kolmogorov complexity: information inequalities

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- submodularity:

  $\mathrm{C}(\mathbf{x},\mathbf{y},\mathbf{z}) + \mathrm{C}(\mathbf{z}) \le \mathrm{C}(\mathbf{x},\mathbf{z}) + \mathrm{C}(\mathbf{y},\mathbf{z}) + O(\log(|\mathbf{x}| + |\mathbf{y}|))$,
  cf. $\mathrm{I}(\mathbf{x} : \mathbf{y} \mid \mathbf{z}) \ge -O(\log(|\mathbf{x}| + |\mathbf{y}| + |\mathbf{z}|))$

Kolmogorov complexity: information inequalities

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Theorem

The same classes of linear inequalities are true for Shannon entropy and (up to a log*-term) for Kolmogorov complexity.*

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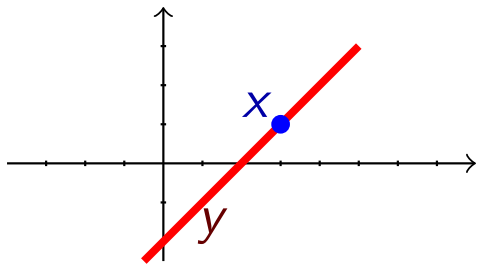

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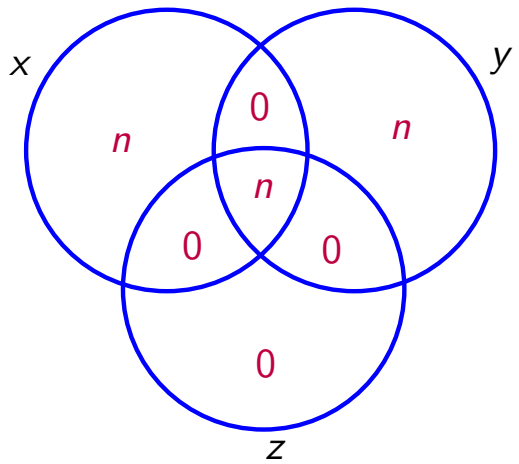
does there exist a z as follows?

- $\mathrm{C}(\mathbf{z}) \approx n$
- $\mathrm{C}(\mathbf{z} \mid \mathbf{x}) \approx 0$
- $\mathrm{C}(\mathbf{z} \mid \mathbf{y}) \approx 0$



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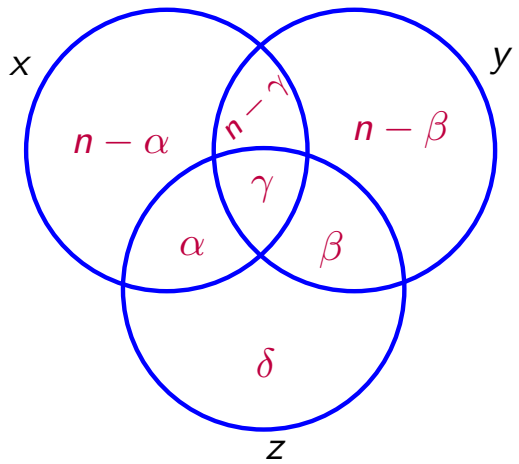
In general, what **profiles** for $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ can we get for these $(\mathbf{x}, \mathbf{y})$ and various z?



secret key agreement

Alice holds **x**

Bob holds **y**

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Alice holds $\mathbf{x} = x_1 \dots x_{2n}$

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Alice and Bob communication via a public channel and agree on a secret key $\mathbf{w}$

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$$\mathrm{prob}[x_i = 1] = 0.5,\ \mathrm{prob}[y_i = 1] = 0.5,\ \mathrm{prob}[x_i = y_i] \approx 0.11$$

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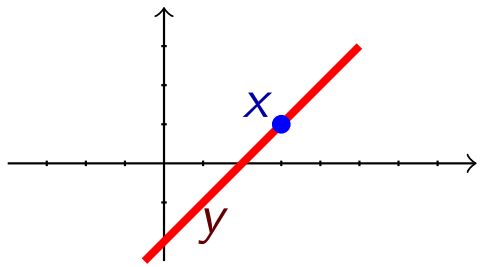

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There exists a protocol that guarantees a secret key **w** *such that* $\mathrm{C}(\mathbf{w}) \approx n$.

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Theorem

- *There exists a protocol with* $\mathrm{C}(\mathbf{w}) \approx n$ *where*
  Alice sends $\mathrm{C}(\mathbf{x} \mid \mathbf{y}) \approx n$ *bits and Bob sends nothing.*
- *There exists a protocol with* $\mathrm{C}(\mathbf{w}) \approx n$ *where*
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- *for some* $\mathbf{x}, \mathbf{y}$ *we cannot do anything substantially different.*

Simple Example: finite field $\mathbb{F}$ with 2^n elements

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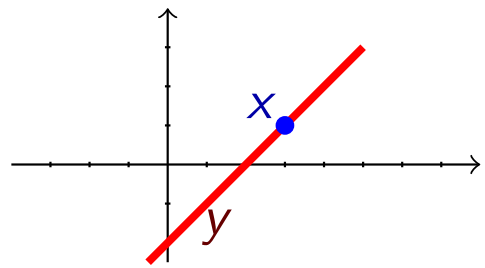

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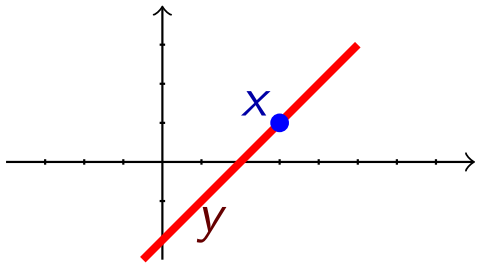

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Theorem

If $\mathrm{C}(\mathbf{w}) \approx n$ *then Alice sends* $\approx n$ *bit or Bob sends* $\approx n$ *bits.*

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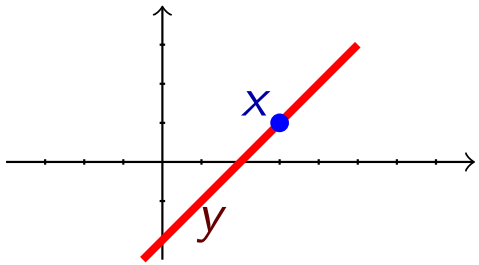

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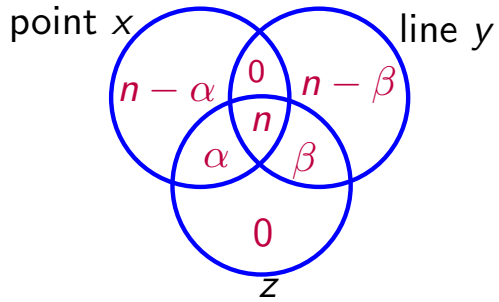
Lemma (An. Muchnik)

In such a **complexity profile** *either $\alpha = n$ or $\beta = n$*



combinatorics of the plane

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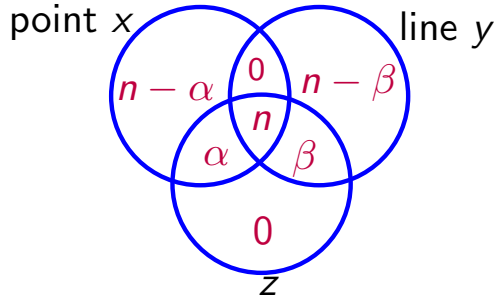

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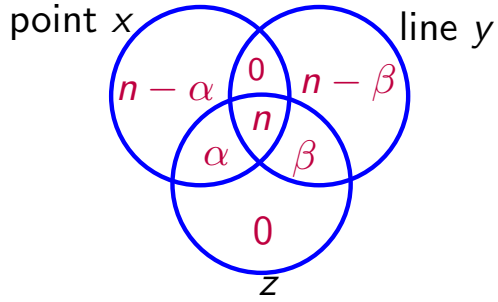

sketch of proof:

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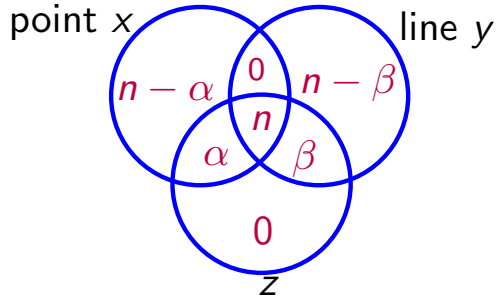

sketch of proof: w.l.o.g. $\alpha \geq \beta$

$$\mathrm{C}(x \mid z) = n - \alpha,\ \mathrm{C}(y \mid z) = n - \beta,\ \mathrm{C}(x, y \mid z) = 2n - \alpha - \beta$$

$$\begin{aligned} A &= \{x' \text{ is a point on } \mathbb{F}^2 \ :\ \mathrm{C}(x' \mid z) \leq n - \alpha\} \\ B &= \{y' \text{ is a line on } \mathbb{F}^2 \ :\ \mathrm{C}(y' \mid z) \leq n - \beta\} \end{aligned}$$

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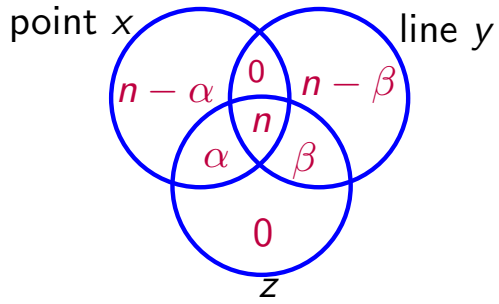

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$$\mathrm{C}(x \mid z) = n - \alpha,\ \mathrm{C}(y \mid z) = n - \beta,\ \mathrm{C}(x, y \mid z) = 2n - \alpha - \beta$$

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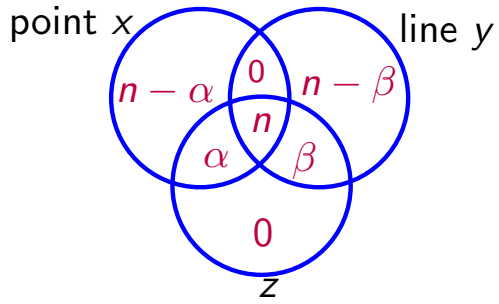
Counting Claim: $\mathrm{Card}(A) = 2^{n-\alpha \pm O(\log n)}$ and $\mathrm{Card}(B) = 2^{n-\beta \pm O(\log n)}$

combinatorics of the plane

Lemma (An. Muchnik)

In such a **complexity profile**
either $\alpha = n$ *or* $\beta = n$



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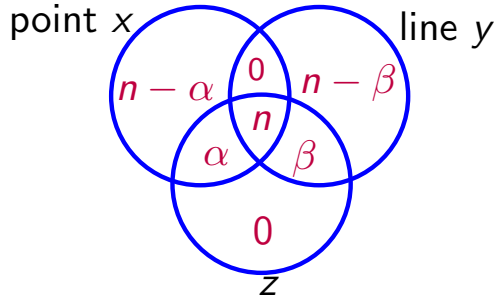

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therefore:

$$\begin{array}{rcl} n - \alpha + n - \beta & = & \\ \mathrm{C}(x, y \mid z) & \leq & \log[\text{ no. of incidences in } A \times B\ ] \\ & \leq & 0.5 \log \mathrm{Card}(A) + \log \mathrm{Card}(B) = 1.5n - 0.5\alpha - \beta \end{array}$$

from combinatorics of the plane to secret sharing

Reminder: a finite plane; $A =$ *some set of lines* and $B =$ *some set of points*

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Comment: some subtler properties of planes imply some subtler bounds for communication protocols

Proceed with the exercises!