Different Views of Information
Part I — supplementary materials

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Geometry in Cryptography and Communication

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UiT The Arctic University of Norway, October 2025

definitions (1)

entropy of random variable X with a distribution $(p_1, \ldots, p_k)$ such that $\sum p_i = 1$

$$\mathrm{H}(X) := \sum p_i \log \frac{1}{p_i}$$

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Conclusion: $\sum_{i=1}^{k} p_i \log \frac{1}{kp_i} \le \log \sum_{i=1}^{k} \left(p_i \frac{1}{kp_i} \right) = \log \sum_{i=1}^{k} \left(\frac{1}{k} \right)$

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Claim: $\mathrm{H}(X)$ can be any real number between 0 and $\log k$

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Claim: $\mathrm{H}(X)$ can be any real number between 0 and $\log k$

i.e., **entropy of a random variable can be any non-negative real number**

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Constraints: $0 \leq \mathrm{H}(X), \mathrm{H}(Y) \leq \mathrm{H}(X, Y) \leq \mathrm{H}(X) + \mathrm{H}(Y)$

basic inequalities

Proposition. $\mathrm{H}(X, Y) \leq \mathrm{H}(X) + \mathrm{H}(Y)$

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Proof:
$p_{ij} := \mathrm{prob}[X = a_i \text{ and } Y = b_j]$, $p_{i*} := \mathrm{prob}[X = a_i]$, $p_{*j} := \mathrm{prob}[Y = b_j]$

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$$= \log \left(\sum_{i} p_{i*} \cdot \sum_{j} p_{*j} \right) = \log 1 = 0$$

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each entropic vector (h_X, h_Y, h_{XY}) satisfies these inequalities

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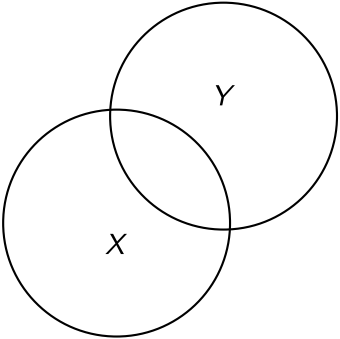
three coordinates, $2 + 2 + 1$ inequalities

Inequalities in a diagram



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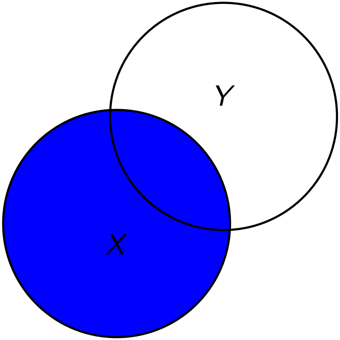

$\mathrm{H}(X)$

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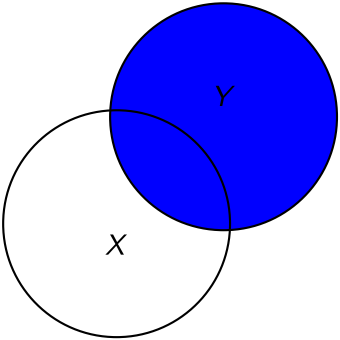
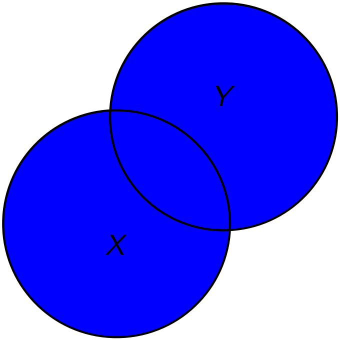


$\mathrm{H}(Y)$

Inequalities in a diagram



$\mathrm{H}(X, Y)$

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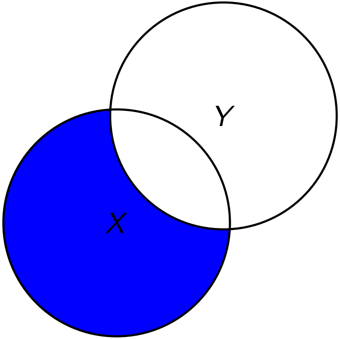
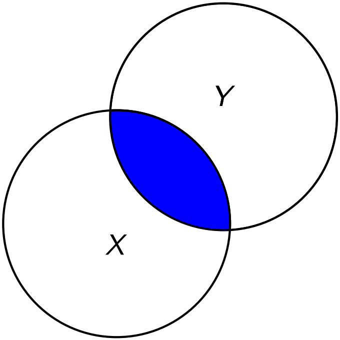


$\mathrm{H}(X \mid Y) = \mathrm{H}(X, Y) - \mathrm{H}(Y)$

Inequalities in a diagram



$\mathrm{I}(X : Y) = \mathrm{H}(X) + \mathrm{H}(Y) - \mathrm{H}(X, Y)$

The definitions (2) again

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$$0 \leq \mathrm{H}(X), \mathrm{H}(Y) \leq \mathrm{H}(X, Y) \leq \mathrm{H}(X) + \mathrm{H}(Y)$$

each entropic vector (h_A, h_B, h_{AB}) satisfies these inequalities

three coordinates, $2 + 2 + 1$ inequalities

~~**Theorem**~~ **Simple observation:** no other inequalities for entropy vector of **pairs**

The definitions (3)

entropic profile (entropic vector) of a **triple** of random variables (X, Y, Z) is

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Inequalities:

- monotonicity, e.g., $\mathrm{H}(X, Y) \leq \mathrm{H}(X, Y, Z)$
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- monotonicity, e.g., $\mathrm{H}(X, Y) \leq \mathrm{H}(X, Y, Z)$
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- subadditivity, e.g., $\mathrm{H}(X, Y) \leq \mathrm{H}(X) + \mathrm{H}(Y)$

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Inequalities:

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each entropic vector (7-tuple of real numbers) satisfies these inequalities

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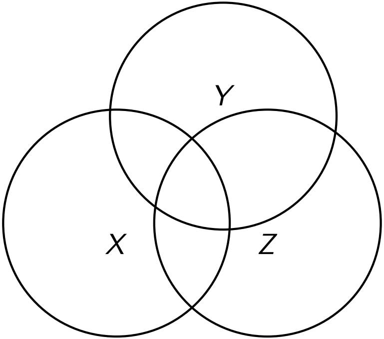
- submodularity, $\mathrm{H}(X, Y, Z) + \mathrm{H}(X) \leq \mathrm{H}(X, Y) + \mathrm{H}(X, Z)$
  (a.k.a. $\mathrm{I}(Y : Z \,|\, X) \geq 0$)

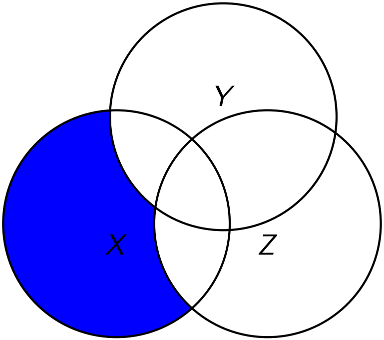
each entropic vector (7-tuple of real numbers) satisfies these inequalities

seven coordinates and *at least* $3 + 3 + 3$ inequalities

Entropy inequalities in diagrams

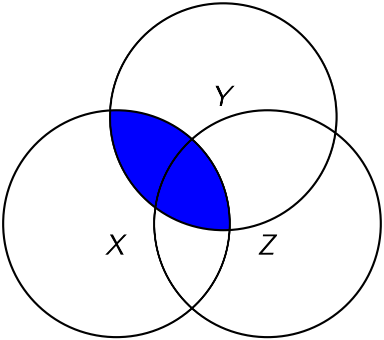


Entropy inequalities in diagrams



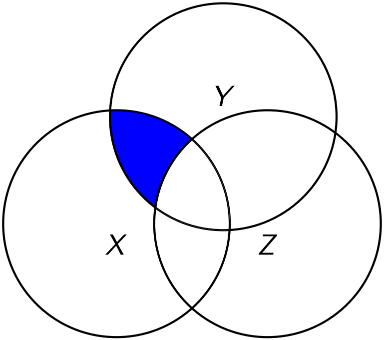
$\mathrm{H}(X \mid Y, Z) = \mathrm{H}(X, Y, Z) - \mathrm{H}(Y, Z)$

Entropy inequalities in diagrams



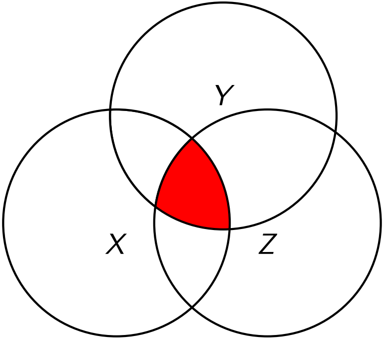
$\mathrm{I}(X : Y) = \mathrm{H}(X) + \mathrm{H}(Y) - \mathrm{H}(X, Y)$

Entropy inequalities in diagrams



$\mathrm{I}(X : Y \mid Z) = \mathrm{H}(X, Z) + \mathrm{H}(Y, Z) - \mathrm{H}(X, Y, Z) - \mathrm{H}(Z)$

Entropy inequalities in diagrams



$\mathrm{I}(X : Y : Z) =$
$\mathrm{H}(X) + \mathrm{H}(Y) + \mathrm{H}(Z) - \mathrm{H}(X, Y) - \mathrm{H}(X, Z) - \mathrm{H}(Y, Z) + \mathrm{H}(X, Y, Z)$

(may be negative!)

The definitions (3) again

entropic profile (entropic vector) of a **triple** of random variables (X, Y, Z) is

$$\langle\ \mathrm{H}(X), \mathrm{H}(Y), \mathrm{H}(Y), \mathrm{H}(X, Y), \mathrm{H}(X, Z), \mathrm{H}(Y, Z), \mathrm{H}(X, Y, Z)\ \rangle$$

or equivalently

$$\begin{array}{lll} \langle\ \mathrm{H}(X \,|\, Y, Z), & \mathrm{H}(Y \,|\, X, Z), \quad \mathrm{H}(Z \,|\, X, Y), & \\ & \mathrm{I}(X : Y \,|\, Z), \quad \mathrm{I}(X : Z \,|\, Y), \quad \mathrm{I}(Y : Z \,|\, X), & \mathrm{I}(X : Y : Z)\ \rangle \end{array}$$

Inequalities:

The definitions (3) again

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Inequalities:

- monotonicity, e.g., $\mathrm{H}(X, Y) \leq \mathrm{H}(X, Y, Z)$ (a.k.a. $\mathrm{H}(Z \mid X, Y) \geq 0$)
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- submodularity, $\mathrm{H}(X, Y, Z) + \mathrm{H}(X) \leq \mathrm{H}(X, Y) + \mathrm{H}(X, Z)$ (a.k.a. $\mathrm{I}(Y : Z \mid X) \geq 0$)

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The definitions (3) one more time

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$$\begin{aligned}\langle\ \mathrm{H}(X \mid Y,Z),\ & \mathrm{H}(Y \mid X,Z), \quad \mathrm{H}(Z \mid X,Y),\\ & \mathrm{I}(X:Y \mid Z), \quad \mathrm{I}(X:Z \mid Y), \quad \mathrm{I}(Y:Z \mid X),\ \mathrm{I}(X:Y:Z)\ \rangle\end{aligned}$$

Observation 1: (vector of entropies) $\leftrightarrow$ (vector of atomic I-measures)
is a non-degenerate linear transformation

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Observation 1: (vector of entropies) $\leftrightarrow$ (vector of atomic I-measures)
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Observation 2: every 7-vector with non-negative atomic I-measures is entropic.

The definitions (3) one more time

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Observation 1: (vector of entropies) $\leftrightarrow$ (vector of atomic I-measures) is a non-degenerate linear transformation

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The definitions (3) one more time

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Observation 1: (vector of entropies) $\leftrightarrow$ (vector of atomic I-measures) is a non-degenerate linear transformation

Observation 2: every 7-vector with non-negative atomic I-measures is entropic. This is a full dimension cone of entropic vectors!

But there are also entropic vectors with a negative value of $\mathrm{I}(X : Y : Z)$.

Fun with diagrams (1)

How to prove $2\mathrm{H}(X, Y, Z) \leq \mathrm{H}(X, Y) + \mathrm{H}(X, Z) + \mathrm{H}(Y, Z)$?

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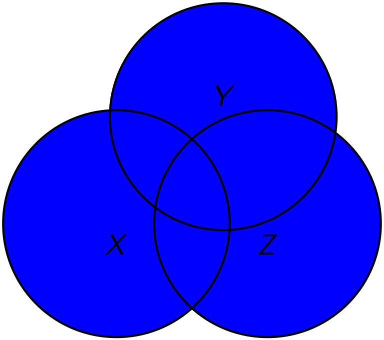

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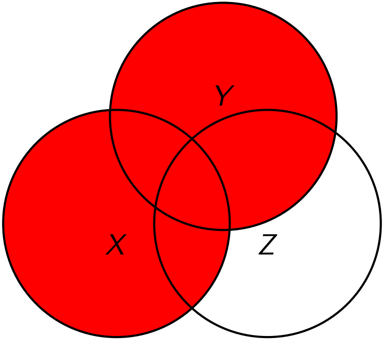

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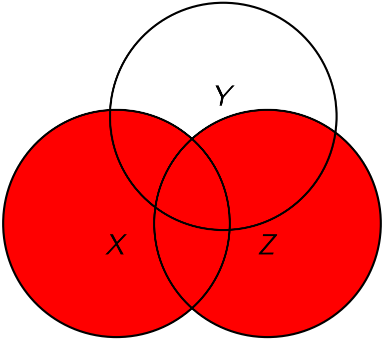

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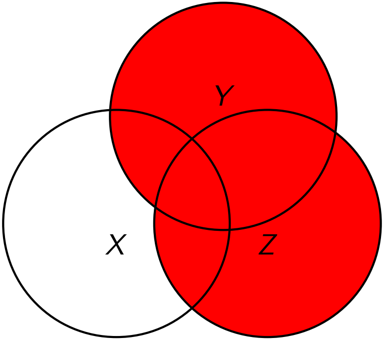
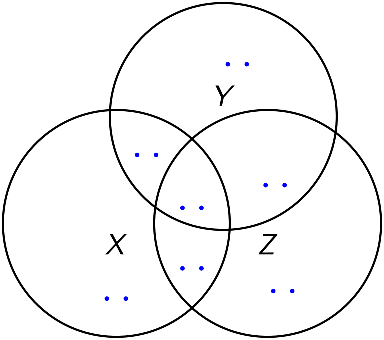


Fun with diagrams (1)

How to prove $\color{blue}{2\mathrm{H}(X, Y, Z)} \leq \mathrm{H}(X, Y) + \mathrm{H}(X, Z) + \mathrm{H}(Y, Z)$?



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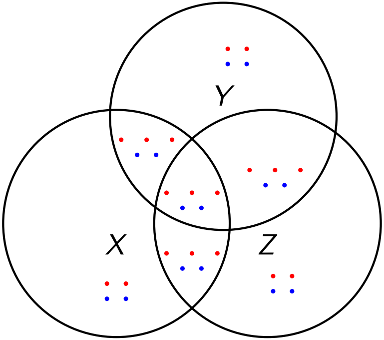

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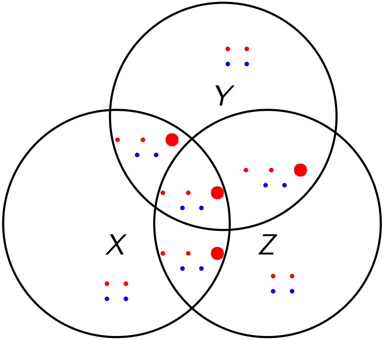

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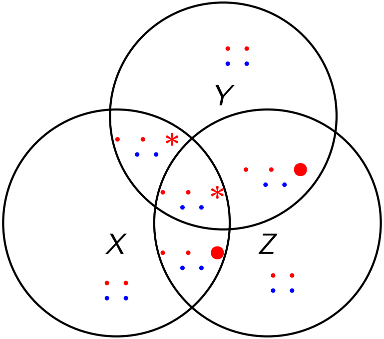
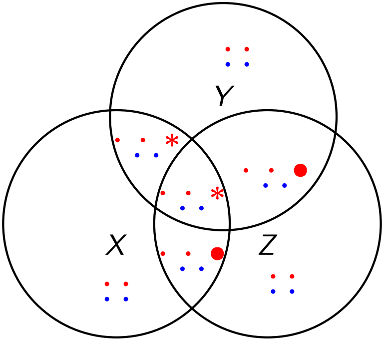


$[\text{RHS}] - [\text{LHS}] = \mathrm{I}(X : Y) + \mathrm{I}(X : Z \mid Y) + \mathrm{I}(Y : Z \mid X)$

Fun with diagrams (1)

How to prove $2\mathrm{H}(X, Y, Z) \leq \mathrm{H}(X, Y) + \mathrm{H}(X, Z) + \mathrm{H}(Y, Z)$?



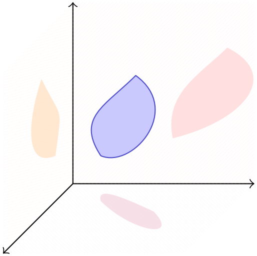
$[\text{RHS}] - [\text{LHS}] = \mathrm{I}(X : Y) + \mathrm{I}(X : Z \mid Y) + \mathrm{I}(Y : Z \mid X) \geq 0$

Fun with diagrams (1)

$2\mathrm{H}(X, Y, Z) \leq \mathrm{H}(X, Y) + \mathrm{H}(X, Z) + \mathrm{H}(Y, Z)$
implies that for every body in $\mathbb{R}^3$

$$\mathsf{Vol}^2 \leq \mathsf{area}(\mathsf{Proj}_{12}) \cdot \mathsf{area}(\mathsf{Proj}_{13}) \cdot \mathsf{area}(\mathsf{Proj}_{23})$$

(the Loomis–Whitney inequality)



Fun with diagrams (2)

If

- $\mathrm{H}(\text{ciphertext} \mid \text{message}, \text{key}) = 0$
- $\mathrm{H}(\text{message} \mid \text{ciphertext}, \text{key}) = 0$
- $\mathrm{I}(\text{message} : \text{ciphertext}) = 0$,

then $\mathrm{H}(\text{key}) \geq \mathrm{H}(\text{message})$

Fun with diagrams (2)

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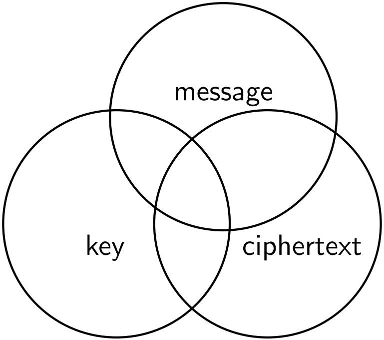

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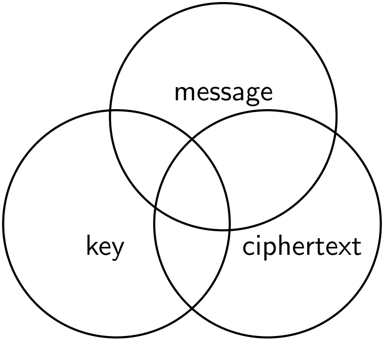

Prove this claim using the diagram!