Different Views of Information
Part I

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Geometry in Cryptography and Communication

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UiT The Arctic University of Norway, October 2025

measure of information: Hartley's approach

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Observation 3: Let $B =$ set of strings of length N in the alphabet $\{a, b, \ldots, z\}$, with 12.7% letters 'e', 9.1% letters 't', 8.2% letters 'a', 7.5% letters 'o', ... Then

$$\chi(B) \approx 4.14N$$

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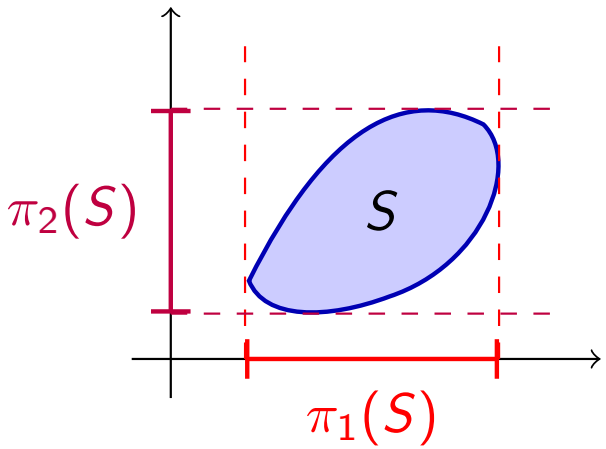

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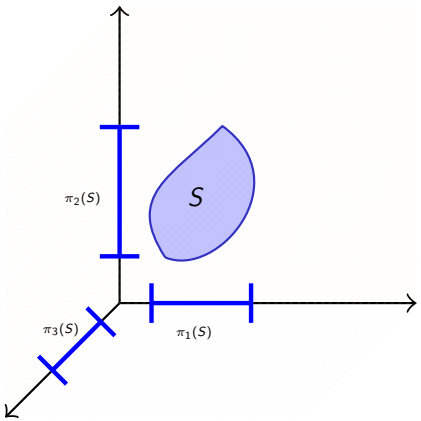

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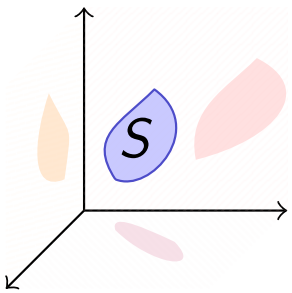
Theorem *(Loomis–Whitney)* If $S \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$, then

$$2 \cdot \chi(S) \leq \chi_{12}(S) + \chi_{23}(S) + \chi_{13}(S).$$



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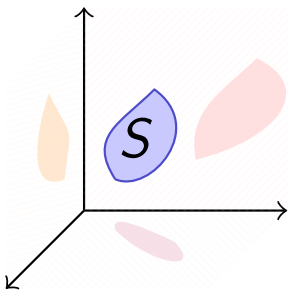

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This inequality basically claims that

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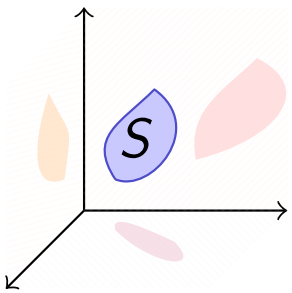

$$2 \cdot \chi(S) \leq \chi_{12}(S) + \chi_{23}(S) + \chi_{13}(S).$$

Continuous version:

$$\mathrm{volume(S)}^2 \leq \mathrm{area}(\pi_{12} S) \cdot \mathrm{area}(\pi_{13} S) \cdot \mathrm{area}(\pi_{23} S).$$

Hartley's information: toy applications

Question 1: There are 25 identical-looking coins, one of which is counterfeit and lighter than the others. Using a balance scale without additional weights, determine the least number of weighings needed to identify the counterfeit coin.

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$$\chi(\text{outcome}) \leq \chi(1\text{st weighing}) + \ldots + \chi(k\text{st weighing}) \leq \underbrace{\log 3 + \ldots + \log 3}_{k}$$

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This is an exercise!

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  (proof: concavity of the logarithm + Jensen's inequality).

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two classical theorems:

- for all uniquely decodable binary code $c_1, \ldots, c_k$

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slightly informally: the value of $\mathrm{H}(X)$ gives the optimal compression rate

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with the usual convention $0 \cdot \log \frac{1}{0} = 0$

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Definition: conditional mutual information between X and Y given Z

1st definition: $\mathrm{I}(X : Y \mid Z) := \sum_c \mathrm{I}(X : Y \mid Z = c) \Pr[Z = c]$,

2nd definition: $\mathrm{I}(X : Y \mid Z) := \mathrm{H}(Y \mid Z) - \mathrm{H}(Y \mid X, Z)$.

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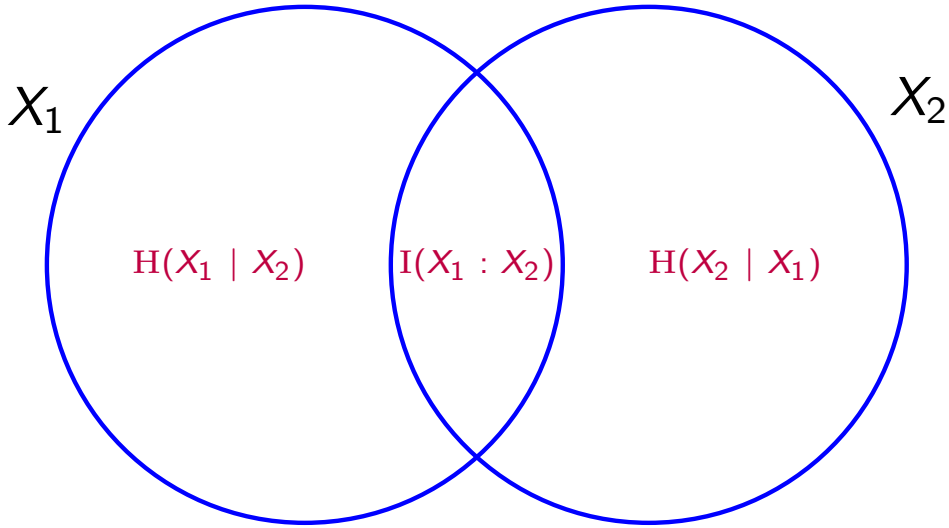

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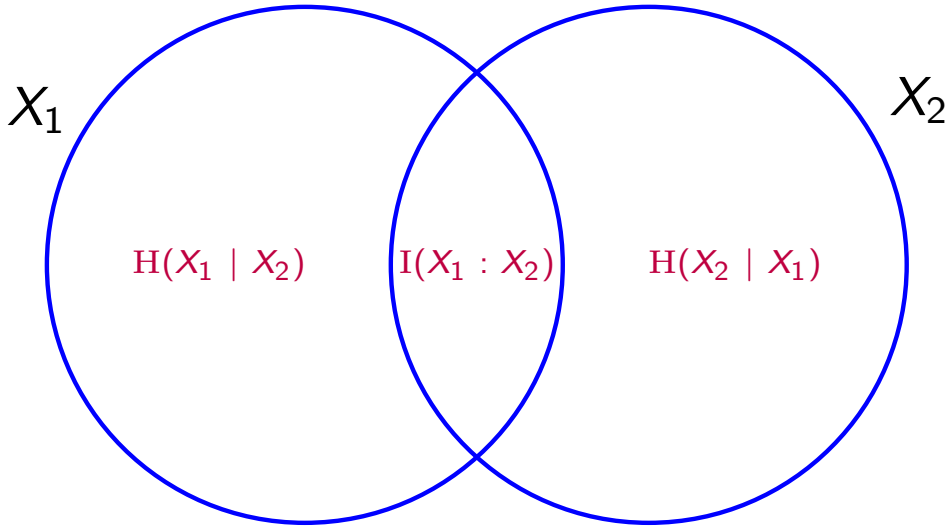

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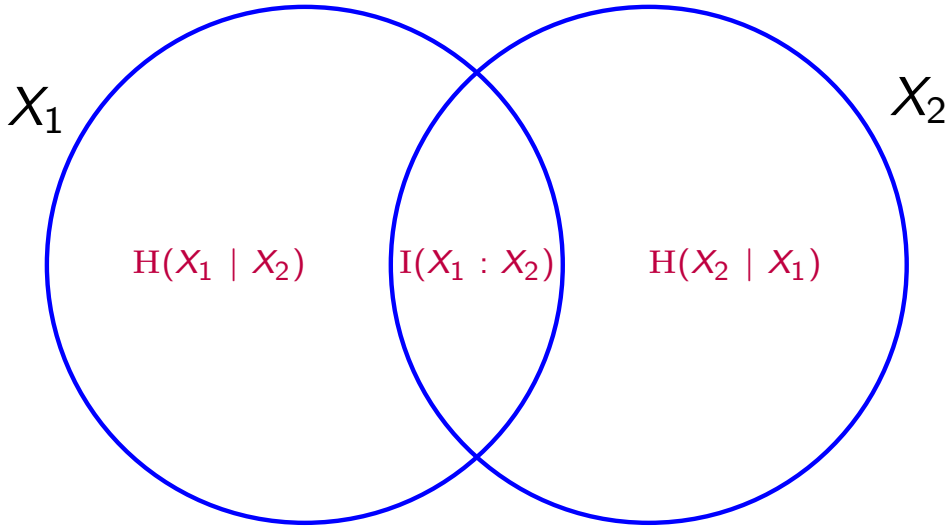

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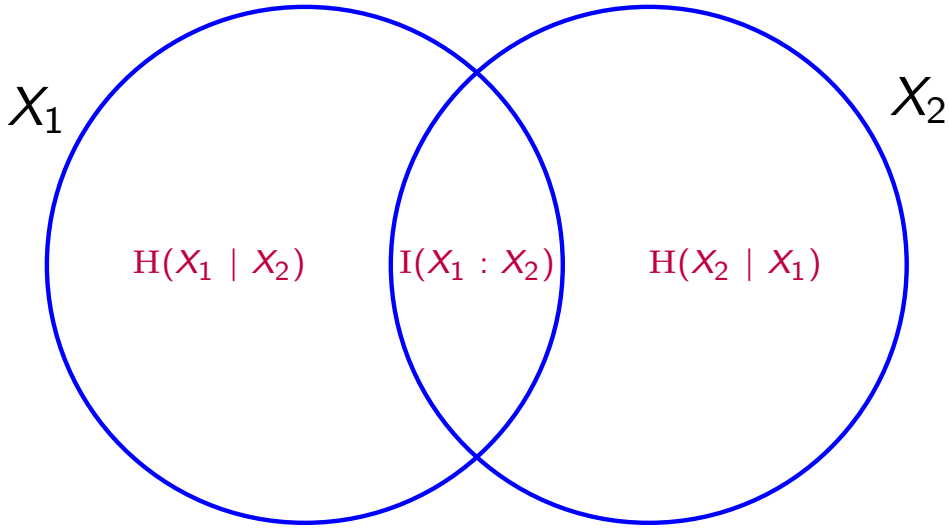

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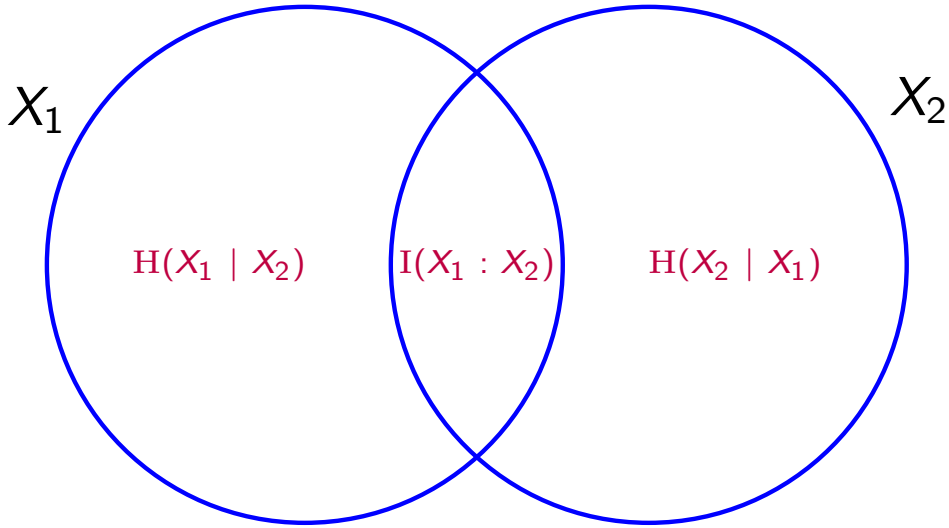

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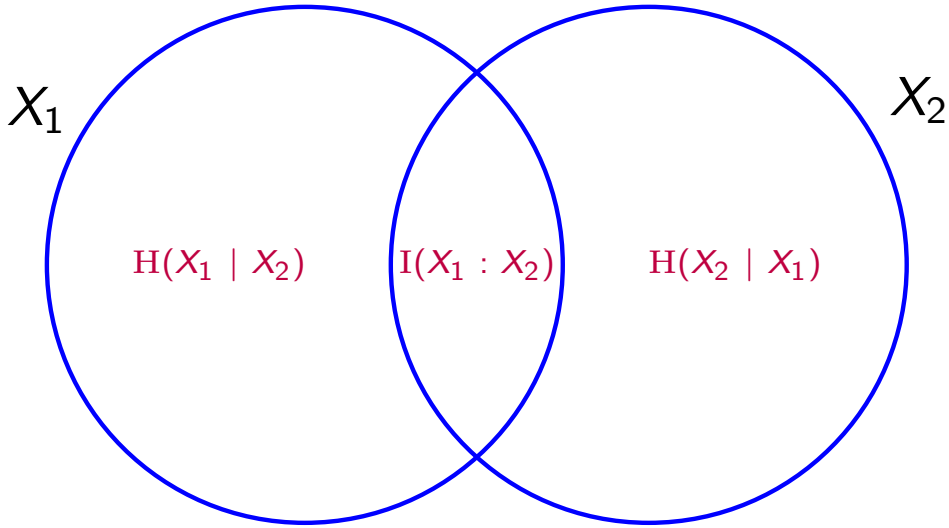

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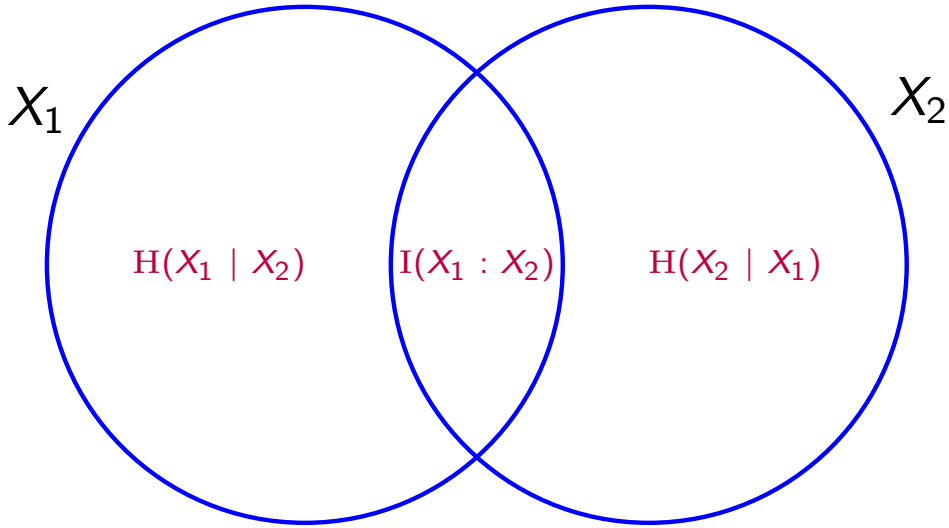

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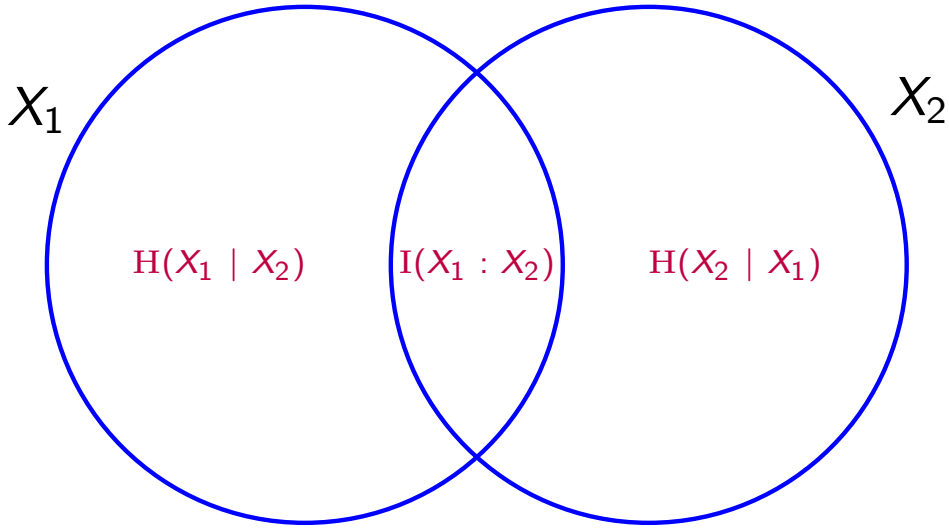
$\mathrm{H}(X_1 \mid X_2),\ \mathrm{H}(X_2 \mid X_1)$

$\mathrm{I}(X_1 : X_2)$



but only **3 parameters** are enough

e.g. $\mathrm{H}(X_1), \mathrm{H}(X_2), \mathrm{H}(X_1, X_2)$ allow to find $\mathrm{H}(X_1 \mid X_2)$, $\mathrm{H}(X_2 \mid X_1)$, $\mathrm{I}(X_1 : X_2)$

or $\mathrm{H}(X_1 \mid X_2)$, $\mathrm{H}(X_2 \mid X_1)$ and $\mathrm{I}(X_1 : X_2)$ allow to find $\mathrm{H}(X_1), \mathrm{H}(X_2), \mathrm{H}(X_1, X_2)$.

constraints: $0 \leq \mathrm{H}(X_1),\ \mathrm{H}(X_2) \leq \mathrm{H}(X_1, X_2) \leq \mathrm{H}(X_1) + \mathrm{H}(X_2)$

equivalently: $\mathrm{H}(X_1 \mid X_2) \geq 0,\ \mathrm{H}(X_2 \mid X_1) \geq 0,\ \mathrm{I}(X_1 : X_2) \geq 0$

Shannon entropy: *entropic profiles*

* For jointly (X_1, X_2, X_3) we have

$\mathrm{H}(X_1),\ \mathrm{H}(X_2),\ \mathrm{H}(X_3)$

$\mathrm{H}(X_1, X_2),\ \mathrm{H}(X_1, X_3),\ \mathrm{H}(X_2, X_3),$

$\mathrm{H}(X_1, X_2, X_3),$

$\mathrm{H}(X_1 \mid X_2),\ \mathrm{H}(X_2 \mid X_1), \ldots,$

$\mathrm{H}(X_1 \mid X_2, X_3), \ldots,$

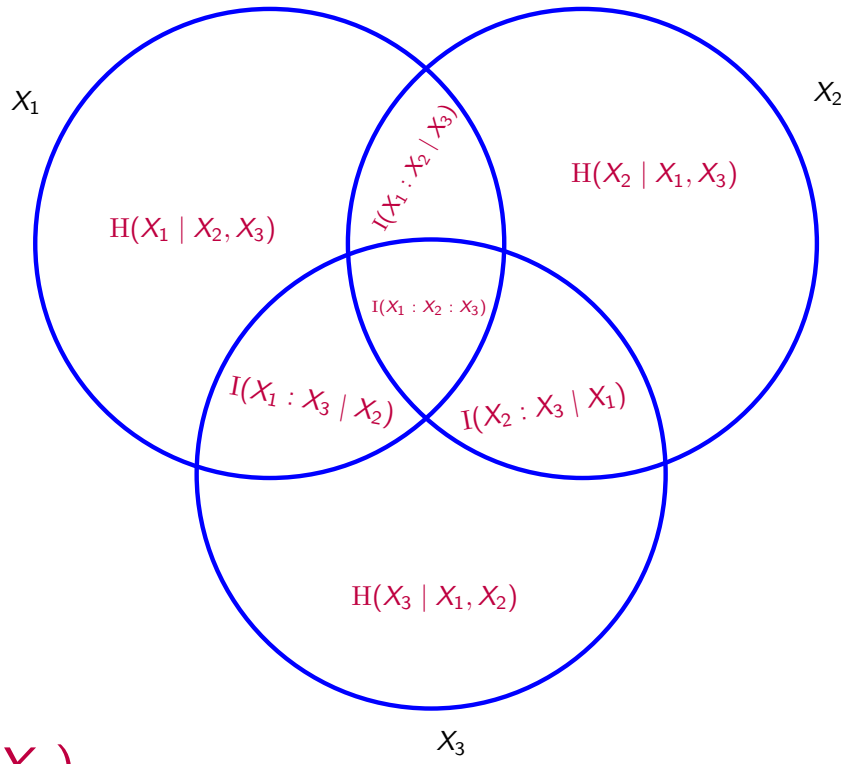
$\mathrm{H}(X_1, X_2 \mid X_3), \ldots,$

$\mathrm{I}(X_1 : X_2),\ \mathrm{I}(X_1 : X_3),\ \mathrm{I}(X_2 : X_3)$

$\mathrm{I}(X_1 : X_2 X_3),\ \mathrm{I}(X_2 : X_1 X_3),\ \mathrm{I}(X_3 : X_1 X_3)$

$\mathrm{I}(X_1 : X_2 \mid X_3),\ \mathrm{I}(X_1 : X_3 \mid X_2),\ \mathrm{I}(X_2 : X_3 \mid X_1)$

$\mathrm{I}(X_1 : X_2 : X_3)$



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$\mathrm{H}(X_1, X_2, X_3),$

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$\mathrm{H}(X_1 \mid X_2, X_3), \ldots,$

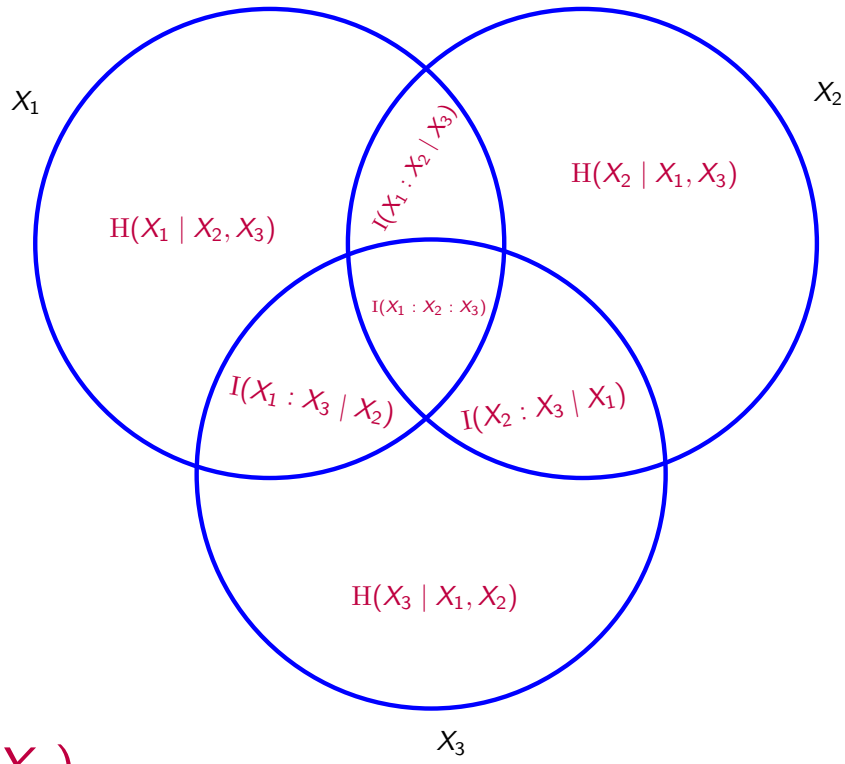
$\mathrm{H}(X_1, X_2 \mid X_3), \ldots,$

$\mathrm{I}(X_1 : X_2),\ \mathrm{I}(X_1 : X_3),\ \mathrm{I}(X_2 : X_3)$

$\mathrm{I}(X_1 : X_2 X_3),\ \mathrm{I}(X_2 : X_1 X_3),\ \mathrm{I}(X_3 : X_1 X_3)$

$\mathrm{I}(X_1 : X_2 \mid X_3),\ \mathrm{I}(X_1 : X_3 \mid X_2),\ \mathrm{I}(X_2 : X_3 \mid X_1)$

$\mathrm{I}(X_1 : X_2 : X_3)$



but only **7 parameters** are enough:

Shannon entropy: *entropic profiles*

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$\mathrm{H}(X_1),\ \mathrm{H}(X_2),\ \mathrm{H}(X_3)$

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$\mathrm{H}(X_1 \mid X_2, X_3), \ldots,$

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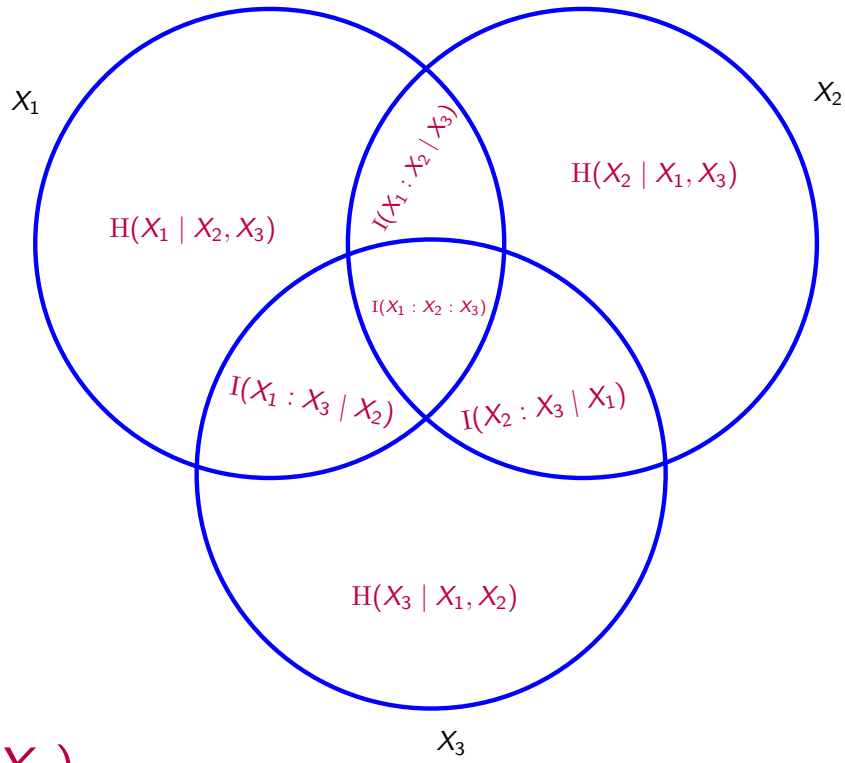
$\mathrm{I}(X_1 : X_2 : X_3)$



but only **7 parameters** are enough:

e.g. $\mathrm{H}(X_1), \mathrm{H}(X_2), \ldots$ allow to find all other quantities

Shannon entropy: *entropic profiles*

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$\mathrm{H}(X_1, X_2, X_3),$

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$\mathrm{H}(X_1 \mid X_2, X_3), \ldots,$

$\mathrm{H}(X_1, X_2 \mid X_3), \ldots,$

$\mathrm{I}(X_1 : X_2),\ \mathrm{I}(X_1 : X_3),\ \mathrm{I}(X_2 : X_3)$

$\mathrm{I}(X_1 : X_2 X_3),\ \mathrm{I}(X_2 : X_1 X_3),\ \mathrm{I}(X_3 : X_1 X_3)$

$\mathrm{I}(X_1 : X_2 \mid X_3),\ \mathrm{I}(X_1 : X_3 \mid X_2),\ \mathrm{I}(X_2 : X_3 \mid X_1)$

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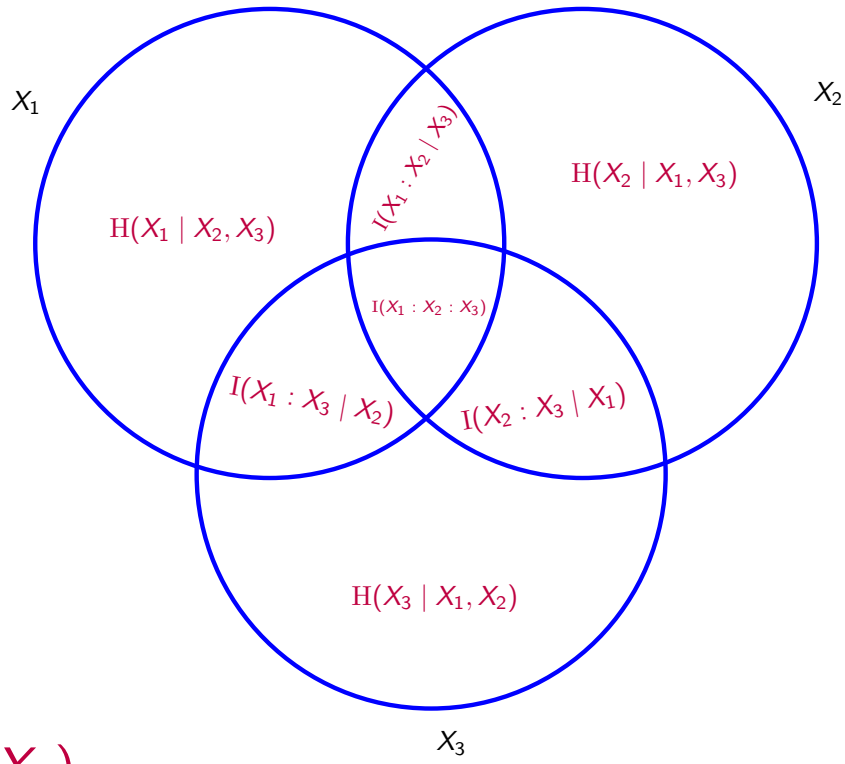

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or $\mathrm{H}(X_1 | X_2, X_3), \ldots, \mathrm{I}(X_1 : X_2 \mid X_3), \ldots, \mathrm{I}(X_1 : X_2 : X_3)$
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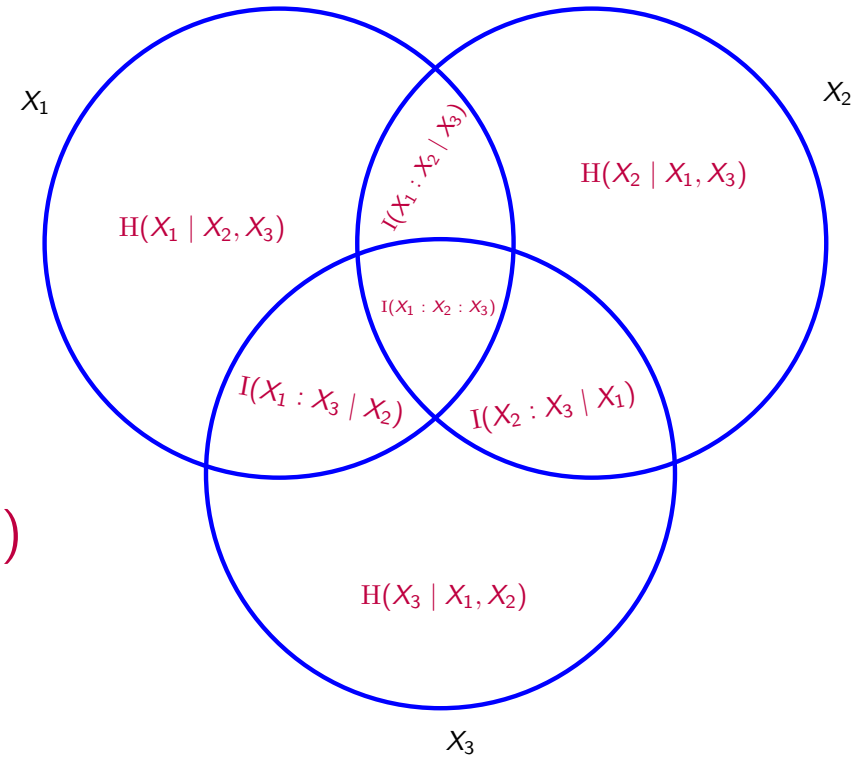
$\mathrm{H}(X_1, X_2, X_3),$

$\mathrm{H}(X_1 \mid X_2, X_3), \ldots,$

$\mathrm{I}(X_1 : X_2 \mid X_3),\ \mathrm{I}(X_1 : X_3 \mid X_2),\ \mathrm{I}(X_2 : X_3 \mid X_1)$

$\mathrm{I}(X_1 : X_2 : X_3)$



7 parameters define the profile:

e.g. $\mathrm{H}(X_1), \mathrm{H}(X_2), \ldots$ allow to find all other quantities

or $\mathrm{H}(X_1|X_2, X_3), \ldots, \mathrm{I}(X_1 : X_2 \mid X_3), \ldots, \mathrm{I}(X_1 : X_2 : X_3)$
allow to find all other quantities

Shannon entropy: *entropic profiles*

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$\mathrm{H}(X_1, X_2, X_3),$

$\mathrm{H}(X_1 \mid X_2, X_3), \ldots,$

$\mathrm{I}(X_1 : X_2 \mid X_3),\ \mathrm{I}(X_1 : X_3 \mid X_2),\ \mathrm{I}(X_2 : X_3 \mid X_1)$

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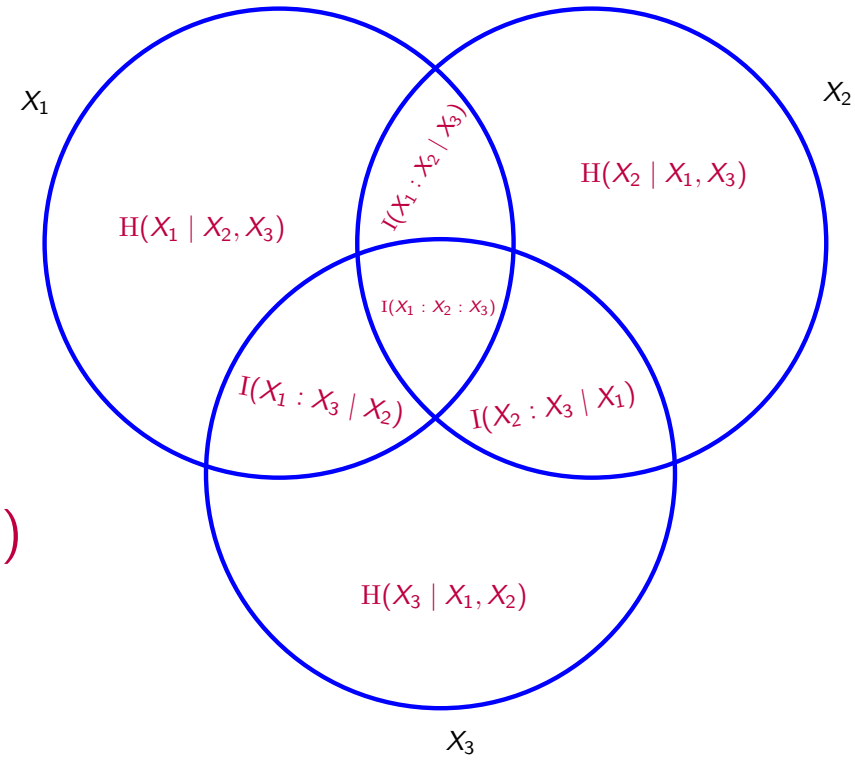

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allow to find all other quantities

9 constraints: $\mathrm{H}(X_1|X_2, X_3) \geq 0, \ldots, \mathrm{I}(X_1 : X_2) \geq 0, \ldots, \mathrm{I}(X_1 : X_2 \mid X_3) \geq 0, \ldots$

Shannon entropy: *entropic profiles*

* For jointly (X_1, X_2, X_3) we have

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$\mathrm{H}(X_1, X_2),\ \mathrm{H}(X_1, X_3),\ \mathrm{H}(X_2, X_3),$

$\mathrm{H}(X_1, X_2, X_3),$

$\mathrm{H}(X_1 \mid X_2, X_3), \ldots,$

$\mathrm{I}(X_1 : X_2 \mid X_3),\ \mathrm{I}(X_1 : X_3 \mid X_2),\ \mathrm{I}(X_2 : X_3 \mid X_1)$

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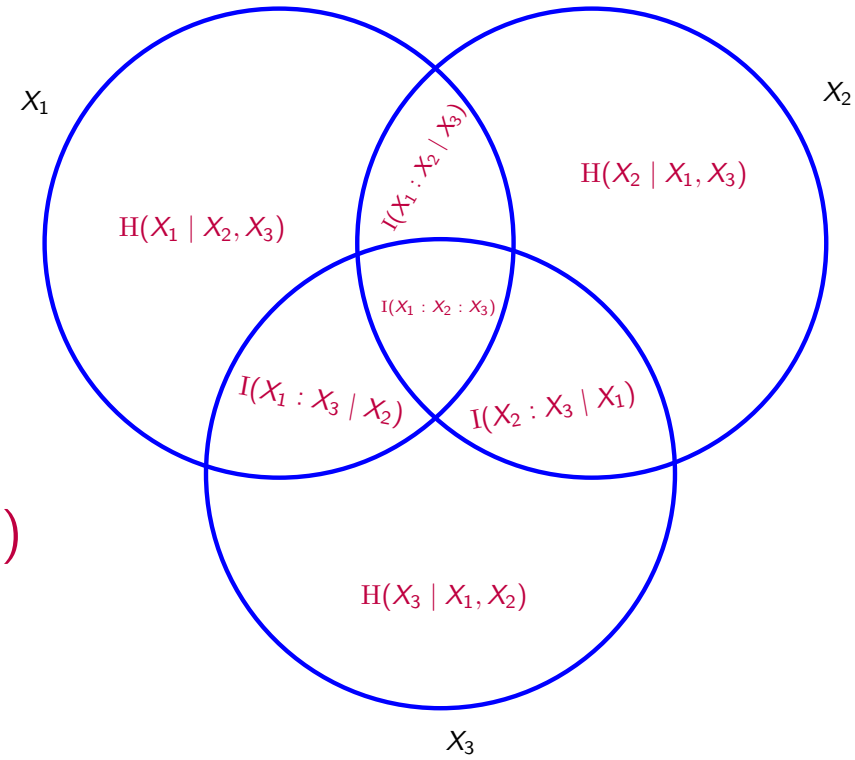

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allow to find all other quantities

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Exercise: no other inequalities for entropies of (X_1, X_2, X_3)

Shannon entropy: *entropic profiles*

* For jointly distributed $(X_1, \ldots, X_n)$ we have

$\mathrm{H}(X_1),\ \mathrm{H}(X_2), \ldots$
$\mathrm{H}(X_1, X_2),\ \mathrm{H}(X_1, X_3), \ldots,$
$\mathrm{H}(X_1, X_2, X_3), \ldots$
$\ldots$

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$\mathrm{H}(X_1, X_2, X_3), \ldots$
$\ldots$

$2^n - 1$ **parameters** that define the profile

Shannon entropy: *entropic profiles*

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$\mathrm{H}(X_1, X_2, X_3), \ldots$
$\ldots$

$2^n - 1$ **parameters** that define the profile

classical constraints:

Shannon entropy: *entropic profiles*

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$\mathrm{H}(X_1, X_2),\ \mathrm{H}(X_1, X_3), \ldots,$
$\mathrm{H}(X_1, X_2, X_3), \ldots$
$\ldots$

$2^n - 1$ **parameters** that define the profile

classical constraints:

- monotonicity:
  $\mathrm{H}(X_{i_1} \ldots X_{i_m}) \leq \mathrm{H}(X_{i_1} \ldots X_{i_m} X_{j_1} \ldots X_{j_s})$

Shannon entropy: *entropic profiles*

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$\ldots$

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classical constraints:

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  $\mathrm{H}(X_{i_1} \ldots X_{i_m}) \leq \mathrm{H}(X_{i_1} \ldots X_{i_m} X_{j_1} \ldots X_{j_s})$
- concavity:
  $\mathrm{H}(X_{i_1} \ldots X_{i_m} X_{j_1} \ldots X_{j_s}) \leq \mathrm{H}(X_{i_1} \ldots X_{i_m}) + \mathrm{H}(X_{j_1} \ldots X_{j_s})$

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$\ldots$

$2^n - 1$ **parameters** that define the profile

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  $\mathrm{H}(X_{i_1} \ldots X_{i_m}) \leq \mathrm{H}(X_{i_1} \ldots X_{i_m} X_{j_1} \ldots X_{j_s})$
- concavity:
  $\mathrm{H}(X_{i_1} \ldots X_{i_m} X_{j_1} \ldots X_{j_s}) \leq \mathrm{H}(X_{i_1} \ldots X_{i_m}) + \mathrm{H}(X_{j_1} \ldots X_{j_s})$
- submodularity:
  $$\mathrm{H}(X_{i_1} \ldots X_{i_m} X_{j_1} \ldots X_{j_s} X_{k_1} \ldots X_{k_\ell}) + \mathrm{H}(X_{k_1} \ldots X_{k_\ell}) \leq$$
  $$\leq \mathrm{H}(X_{i_1} \ldots X_{i_m} X_{k_1} \ldots X_{k_\ell}) + \mathrm{H}(X_{j_1} \ldots X_{j_s} X_{k_1} \ldots X_{k_\ell})$$

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$\ldots$

$2^n - 1$ **parameters** that define the profile

classical constraints:

for $\mathcal{M} = \{i_1, \ldots, i_m\}$ we denote $\mathrm{H}(X_{\mathcal{I}}) = \mathrm{H}(X_{i_1} \ldots X_{i_m})$

- monotonicity:
  $\mathrm{H}(X_{\mathcal{I}}) \leq \mathrm{H}(X_{\mathcal{I}\cup\mathcal{J}})$ or equivalently $\mathrm{H}(X_{\mathcal{J}} \mid X_{\mathcal{I}}) \geq 0$
- concavity:
  $\mathrm{H}(X_{\mathcal{I}\cup\mathcal{J}}) \leq \mathrm{H}(X_{\mathcal{I}}) + \mathrm{H}(X_{\mathcal{J}})$ or equivalently $\mathrm{I}(X_{\mathcal{I}} : X_{\mathcal{J}}) \geq 0$
- submodularity:
  $\mathrm{H}(X_{\mathcal{I}\cup\mathcal{J}\cup\mathcal{K}}) + \mathrm{H}(X_{\mathcal{K}}) \leq \mathrm{H}(X_{\mathcal{I}\cup\mathcal{K}}) + \mathrm{H}(X_{\mathcal{J}\cup\mathcal{K}})$
  or equivalently $\mathrm{I}(X_{\mathcal{I}} : X_{\mathcal{J}} \mid X_{\mathcal{K}}) \geq 0.$

Shannon entropy: *entropic profiles*

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$\ldots$

$2^n - 1$ **parameters** that define the profile

Shannon, the 1940s:

- $\mathrm{H}(X_{\mathcal{J}} \mid X_{\mathcal{I}}) \geq 0$
- $\mathrm{I}(X_{\mathcal{I}} : X_{\mathcal{J}}) \geq 0$
- $\mathrm{I}(X_{\mathcal{I}} : X_{\mathcal{J}} \mid X_{\mathcal{K}}) \geq 0$

Shannon entropy: *entropic profiles*

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Question: Are there any other inequalities?

Shannon entropy: *entropic profiles*

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$\mathrm{H}(X_1, X_2, X_3), \ldots$
$\ldots$

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Zhang–Yeung 1998: Yes, another inequality for $n = 4$ random variables!

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Question: Are there any other inequalities?

Zhang–Yeung 1998: Yes, another inequality for $n = 4$ random variables!

Matus 2007: **infinitely many** inequalities with $n \geq 4$ random variables!!

Shannon entropy: *entropic profiles*

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this vector of $2^n - 1$ **reals** $\Rightarrow$ an **entropic profile**

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Notation: $\mathbf{P}_n :=$ entropic profiles for all $(X_1, \ldots, X_n)$

Shannon entropy: *entropic profiles*

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Theorem. $\bar{\mathbf{P}}_n$ (topological closure) is a *convex cone*.

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Exercise: prove it.

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this vector of $2^n - 1$ **reals** $\Rightarrow$ an **entropic profile**

Notation: $\mathbf{P}_n :=$ entropic profiles for all $(X_1, \dots, X_n)$

Theorem. $\bar{\mathbf{P}}_n$ (topological closure) is a *convex cone*.

Exercise: prove it.

Fact:

- the case $n \leq 3$ is simple : $\bar{\mathbf{P}}_1$, $\bar{\mathbf{P}}_2$, $\bar{\mathbf{P}}_3$ are defined by Shannon's inequalities $\mathrm{H}(X_\mathcal{J} \mid X_\mathcal{I}) \geq 0$, $\mathrm{I}(X_\mathcal{I} : X_\mathcal{J}) \geq 0$, $\mathrm{I}(X_\mathcal{I} : X_\mathcal{J} \mid X_\mathcal{K}) \geq 0$
- the case $n \geq 4$ is hard : $\bar{\mathbf{P}}_n$ is not polyhedral

  (and not filly understood)

Shannon entropy: *symmetric key encryption*

setting:

- **Sender** and **Receiver** sample a common secret **key**

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solution: Vernam's scheme / one-time pad,
where **message**, **key**, **ciphertext** $\in \{0,1\}^n$

- **ciphertext** $=$ bitwise XOR of **message** and **key**
- **key** $=$ bitwise XOR of **message** and **ciphertext**

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Theorem (Shannon): $\mathrm{H}(\textbf{key}) \geq \mathrm{H}(\textbf{message})$

Shannon entropy: *perfect secret sharing* (threshold scheme)

setting: fix integer numbers n and $t \leq n$.

- **Dealer** samples a secret key S_0 and shares $S_1, \ldots S_n$
- for $i = 1, \ldots, n$ the i-th **participant** gets the share S_i
- correctness: every t **participants** together can compute S_0
  $\mathrm{H}(S_0 \mid S_{i_1} \ldots S_{i_t}) = 0$
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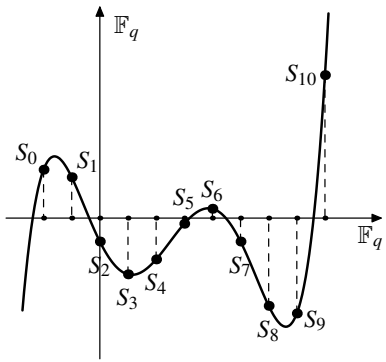

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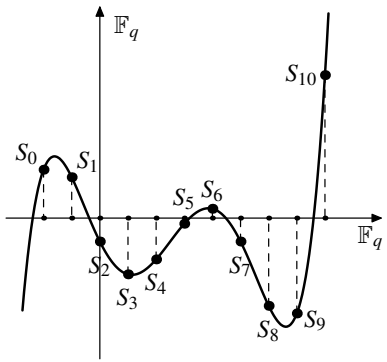

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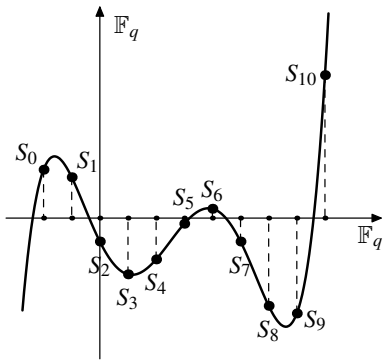

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Exercise: prove that in *every* correct and secure scheme of secret sharing
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- **Dealer** samples a secret key S_0 and shares S_A, S_B, S_C, S_D for 4 participants
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Much simpler fact: for all conditions of correctness and security there exists a secret sharing scheme such that $\mathrm{H}(S_i) \le 2^{O(n)} \cdot \mathrm{H}(S_0)$ for all i