**UiT the Arctic University, October 2025.**
**Fall school on Geometry in Cryptography and Communication.**

**Mini-course *Different Views of Information.* Exercises Part II.**

**Exercise 1.** Use the definition of Kolmogorov complexity and prove formally
— $\mathrm{C}(\mathbf{x}) \le |\mathbf{x}| + O(1)$,
— $\mathrm{C}(\mathbf{xx}) \le |\mathbf{x}| + O(1)$.

**Exercise 2.** Use the definition of Kolmogorov complexity and prove formally that
— for every $n$ there exists a bit string $\mathbf{x}$ of length $n$ with complexity at least $n$,
— there exists a $\lambda$ such that for every $n$, at least 99% of strings of length $n$ satisfy
$n - \lambda \ \le\ \mathrm{C}(\mathbf{x}) \ \le\ n + \lambda$.
— if a bit string $\mathbf{x}$ contains $p_0 n$ zeros and $p_1 n$ ones, then
$\mathrm{C}(\mathbf{x}) \le \log \frac{n!}{(p_0 n)! \cdot (p_1 n)!} + O(\log n) = \left( p_0 \log \frac{1}{p_0} + p_1 \log \frac{1}{p_1} \right) n + O(\log n)$

**Exercise 3.** Prove that for all sufficiently large $n$, every $n$-bit string $\mathbf{x}$ such that $\mathrm{C}(\mathbf{x}) \ge n$ contains as a factor the string 01010001.

**Exercise 4.** Prove that in every $n$-bit string $\mathbf{x}$ such that $\mathrm{C}(\mathbf{x}) \ge n$ there is a factor of all zeros $\underbrace{000\ldots0}_{\log k}$ of length $k \ge \Omega(\log n)$.

**Exercise 5.** Use the definition of conditional Kolmogorov complexity and prove formally
(a) $\mathrm{C}(\mathbf{x} \mid \mathbf{y}) \le \mathrm{C}(\mathbf{x}) + O(1)$,
(b) $\mathrm{C}(\mathbf{x} \mid \mathbf{x}) = O(1)$,
(c) $\mathrm{C}(\mathbf{x}^{-1} \mid \mathbf{x}) = O(1)$, where $\mathbf{x}^{-1}$ is the word $\mathbf{x}$ written in reverse order.

**Exercise 6.** Prove that
(a) the mapping $\mathbf{x} \mapsto \mathrm{C}(\mathbf{x})$ is not a computable function
(i.e., given $\mathbf{x}$, we cannot compute $\mathrm{C}(\mathbf{x})$ algorithmically);
(b) there is no algorithm that, for every input $n \in \mathbb{N}$, produces a string $\mathbf{x}_n$ such that $\mathrm{C}(\mathbf{x}_n) > n$.

**Exercise 7.** Almost all (*all except for finitely many*) statements of the form "$\mathrm{C}(\mathbf{x}) > |\mathbf{x}|/2$" are true but unprovable. (This gives a version of Gödel's *incompleteness* theorem.)

**Exercise 8.** Prove that
(a) $\mathrm{C}(\mathbf{x}, \mathbf{y}) \le \mathrm{C}(\mathbf{x}) + \mathrm{C}(\mathbf{y}) + O(\log(|\mathbf{x}| + |\mathbf{y}|))$
(b) $\mathrm{C}(\mathbf{x}, \mathbf{y}) \le \mathrm{C}(\mathbf{x}) + \mathrm{C}(\mathbf{y} \mid \mathbf{x}) + O(\log(|\mathbf{x}| + |\mathbf{y}|))$
(c) $\mathrm{C}(\mathbf{x}, \mathbf{y}) \ge \mathrm{C}(\mathbf{x}) + \mathrm{C}(\mathbf{y} \mid \mathbf{x}) - O(\log(|\mathbf{x}| + |\mathbf{y}|))$ [Kolmogorov–Levin theorem]

**Exercise 9.** Prove that for all strings $\mathbf{x}, \mathbf{y}, \mathbf{z}$

$$2\mathrm{C}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \le \mathrm{C}(\mathbf{x}, \mathbf{y}) + \mathrm{C}(\mathbf{x}, \mathbf{z}) + \mathrm{C}(\mathbf{y}, \mathbf{z}) + O(\log n),$$

where $n$ is the sum of lengths of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$.

**Exercise 10.** Let $\mathbb{F}_q$ be a finite field with $q$ elements, and let $\mathbb{P}^2(\mathbb{F}_q)$ be the projective plane over this field. Let $(\mathbf{x}, \mathbf{y})$ be an *incidence* on this plane, i.e., $\mathbf{x}$ is a *point* in this plane, and $\mathbf{y}$ is a u*line* that goes through the chosen point.

(a) Prove that $\mathrm{C}(\mathbf{x}) \le 2n + O(\log n), \mathrm{C}(\mathbf{y}) \le 2n + O(\log n), \mathrm{C}(\mathbf{x}, \mathbf{y}) \le 3n + O(\log n)$, where $n = \log q$.

(b) Prove that if $\mathrm{C}(\mathbf{x}, \mathbf{y}) \ge 3n$ (where again $n = \log q$), then

$$\mathrm{C}(\mathbf{x}) = 2n \pm O(\log n), \mathrm{C}(\mathbf{y}) \pm 2n + O(\log n), \mathrm{C}(\mathbf{x} \mid \mathbf{y}) = n \pm O(\log n), \mathrm{C}(\mathbf{y} \mid \mathbf{x}) = n \pm O(\log n).$$

(c) Prove that if $\mathrm{C}(\mathbf{x}, \mathbf{y}) \ge 3n$ (for $n = \log q$), then for any $\mathbf{z}$ of length at most $100n$ we have

$$\mathrm{C}(\mathbf{z}) \le 2 \cdot \mathrm{C}(\mathbf{z} \mid \mathbf{x}) + 2 \cdot \mathrm{C}(\mathbf{z} \mid \mathbf{y}) + O(\log n)$$